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A MATHEMATICAL APPROACH TO THE ANALYSIS
AND DESIGN OF INTERNAL CONTROL SYSTEMS

Barry E. Cushing

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College of Commerce and Business Administration
University of Illinois at Urbana-Champaign

FACULTY WORKING PAPERS

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March 1, 1973

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by

Barry E. Cushing
Visiting Associate Professor of Accountancy
The University of Illinois

A MATHEMATICAL APPROACH
TO THE ANALYSIS AND DESIGN OF INTERNAL CONTROL SYSTEMS

The problems of designing and evaluating internal control systems have concerned accountants and auditors for many years. Until recently little attention has been paid to the possibility of applying mathematical modelling techniques to these problems. Perhaps the closest thing to an analytical technique in this field is the internal control questionnaire in wide use among auditors. However, the nominal measures generated by such questionnaires are of dubious value for purposes of developing comprehensive models of internal control systems. Evidence of this statement is provided by the failure of one such approach to germinate following its exposure over a decade ago.¹

Recent accounting literature has examined mathematically the subject of control in accounting.² However, these presentations have centered around the concept of feedback. While this is certainly a vital management control concept, its usefulness in connection with traditional analysis of internal control is yet to be demonstrated.

¹ Gene Brown, "Objective Internal Control Evaluation," The Journal of Accountancy, November, 1962, pp. 50-56.

² See for example Yuji Ijiri and Gerald L. Thompson, "Applications of Mathematical Control Theory to Accounting and Budgeting (The Continuous Wheat Trading Model)," The Accounting Review, April 1970, pp. 246-258; Nicholas J. Gonedes, "Optimal Timing of Control Messages for a Two-State Markov Process," Journal of Accounting Research, Autumn 1971, pp. 236-252; and Joel S. Demski, Information Analysis (Reading, Mass., Addison-Wesley Publishing Company, Inc., 1972), Chapter Six.

The purposes of this paper are (1) to describe a means of representing internal control in mathematical terms, (2) to demonstrate how such mathematical representations may be useful to controllers and auditors in designing and evaluating internal control systems, and (3) to discuss the implications of this approach for future research in accounting and auditing.

The type of internal controls of concern here are those intended to prevent inaccuracies in data processing or embezzlement. Feedback control systems are not primarily designed to prevent errors or deviations, but to detect and report on their existence in order that some control adjustment may be made. Therefore, the modelling techniques which adopt the concept of feedback are not discussed in this paper.

To clarify the presentation, an example of an accounting process is used throughout the paper. The example involves the posting of cash receipts to customer accounts within an industrial organization. Possible errors in such a system include (1) a discrepancy between the amount of cash actually received and the amount eventually posted to the accounts receivable control account, (2) posting of a receipt to the wrong account, (3) overpayment or underpayment by the customer, and (4) embezzlement of cash receipts. Possible control measures include (1) clerical review and reconciliation of individual receipts to each customer account prior to posting by the accounts receivable clerk, (2) separation of the function of accounts receivable clerk from the function of cashier, (3) checking of control totals by the accounts receivable clerk subsequent to posting of a batch of receipts, (4) comparison of data provided by the

accounts receivable clerk with that provided by the cashier prior to posting to the general ledger by the general ledger clerk, and (5) preparation of a bank reconciliation periodically by an independent authority. Each of these examples of errors and control techniques is used at some point within the paper to illustrate the application of the internal control model.

A distinction important to the model is that between an internal control technique and an internal control system. An internal control technique is basically a single procedure, whereas an internal control system may consist of several internal control techniques. A given internal control technique may be designed to prevent one error only, or possibly two or more errors. Within an internal control system, some of the techniques used may be intended to prevent one error, while others may be intended to prevent a number of different errors. Conversely, any given error may be the focus of one control technique or of two or more control techniques. Since the approach to modelling an internal control system builds upon models of internal control techniques, this paper first describes the latter and then proceeds to the former.

Internal Control Techniques

As mentioned, a single internal control technique may be intended to prevent one error alone or multiple errors. Both of these situations are considered in this section.

Single Control-Single Error Case

A general model of this situation is diagramed in Exhibit 1. The situation consists of a process subject to control, a control procedure which monitors the process, and an error-correction procedure. The use of a control total in cost receipts processing provides a good example. Assume that one clerk prepares a control total of the dollar amount received for all remittances each day, and then provides the remittance advices to a second clerk, who then posts each remittance to the corresponding account. At the completion of posting, the second clerk obtains a total of the updated balances of all accounts. This updated total is subtracted from the control account total prior to posting, and the result is compared to the original control total. If there is no discrepancy, the process is considered complete. If a discrepancy exists, an error correction procedure is initiated for the purpose of detecting and correcting the specific error or errors underlying the discrepancy.

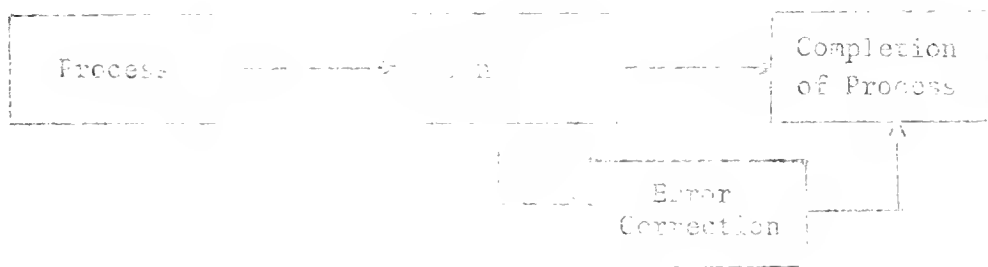


EXHIBIT 1

The mathematical technique used to model this situation is a simple stochastic model drawn from the field of reliability engineering.³

³ For a basic reference on this subject, see David K. Lloyd and Myron Lipow, Reliability: Management, Methods, and Mathematics (Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1962).

Basically, this model provides a means of computing the reliability of a process, which is the probability that the process will be completed with no errors. Reliability in this simple case is a function of the following parameters: (1) the probability, p , that the process is correctly executed prior to administering the control technique, (2) the probability, $P(e)$, that the control step will detect and signal an error given that one exists, (3) the probability, $P(s)$, that the control step will not signal an error given that none exists, (4) the probability, $P(c)$, that the correction step will correct an error given that one exists and has been signaled, and (5) the probability, $P(d)$, that failure of the control will be detected and no correction made given that the control signals an error when none exists.

Reliability, indicated by R , in this case is equal to the sum of (1) $pP(s)$, the probability that the process is executed correctly and the control step does not signal an error, (2) $p(1-P(s))P(d)$, the probability that the process is executed correctly, the control step signals an error, and the failure of the control is detected and no attempt at correction is made, and (3) $(1-p)P(e)P(c)$, the probability that an error in the process is made, but that the control step signals the error and the proper correction is made. Reliability is therefore expressed as:

$$R = pP(s) + p(1-P(s))P(d) + (1-p)P(e)P(c)$$

By a similar process of enumeration of failure possibilities, it can be shown that the probability of a failure to correctly complete the process is

$$1-R = (1-p)(1-P(e)) + (1-p)P(e)(1-P(c)) + p(1-P(s))(1-P(d))$$

The contribution of the control step to this process is to increase the probability that the process will be completed with no error by $R-p$.⁴ To determine whether the control step is worthwhile, however, it is necessary to examine costs. The cost parameters which are necessary to the analysis are (1) C_c , the cost of performing the control procedure each time the process is executed, (2) C_s , the average cost of searching for an error and detecting whether one exists once the control technique has signalled that one exists, and then making whatever corrections are necessary, and (3) C_e , the average cost of an uncorrected error. Given these parameters, it is obvious that the expected total cost of errors in the process if the control step is not performed is

$$C_t = (1-p)C_e$$

However, if the control step is performed, the expected total cost is equal to the sum of the expected costs of (1) performance of the control step, C_c (2) uncorrected errors, $(1-R)C_e$, and (3) search, detection and correction, $[p(1-P(s)) + (1-p)P(s)]C_s$. This total is therefore expressed as

$$C_t' = C_c + (1-R)C_e + [p(1-P(s)) + (1-p)P(s)]C_s$$

The cost differential $C_t' - C_t$ should provide a controller with a useful basis for the choice of whether to use a particular control procedure in a particular situation.

To illustrate these concepts numerically, consider the following set of hypothetical parameter values for the cash receipts processing example:

⁴ Alternatively, the control step could actually decrease the reliability of the process by $p-R$ if $p > R$.

$$\begin{array}{ll}
 p = .8 & P(d) = .99 \\
 P(e) = .95 & C_c = 2 \\
 P(s) = .9 & C_s = 3 \\
 P(c) = .98 & C_e = 20
 \end{array}$$

Table 1 summarizes the computations outlined above. These figures indicate that the use of a batch control total in this example would increase reliability of posting from 80% to over 98% and would reduce average total cost by .898 units per day.

Elements summed to obtain R	Elements summed to obtain (1-R)
$pP(s) = (.8)(.9) = .7200$	$(1-p)(1-P(e)) = (.2)(.05) = .0100$
$p(1-P(s))P(d) = (.8)(.1)(.99) = .0792$	$(1-p)P(e)(1-P(c)) = (.2)(.95)(.02) = .0038$
$(1-p)P(e)P(c) = (.2)(.95)(.98) = .1862$	$P(1-P(s))(1-P(d)) = (.8)(.1)(.01) = .0008$
$R = .9854$	$(1-R) = .0146$
$R - p = .9854 - .8 = .1854$ $C_t = (1-p)C_e = (.2)(20) = 4$ $C_t' = C_c + (1-R)C_e + [p(1-P(s)) + (1-p)P(e)]C_s$ $= 2 + (.0146)20 + (.27)(3)$ $= 3.102$	

Table 1

The reliability model described above illustrates the basic framework of the approach presented in this paper. Within this framework are identified all of the basic parameters which are relevant to the analytical evaluation of internal control. In the next section, this model is extended to illustrate its application in a situation where one control technique may provide control over two or more possible errors.

Single Control-Multiple Error Case

As an example of this situation, consider once again the process of posting cash receipts to an accounts receivable ledger. One control technique which a posting clerk might use is to reconcile the receipt to the charges in the ledger account - that is, to figure out which specific charges the payment is intended. In an industrial setting, most customers will include with payment a remittance advice which provides the data necessary to perform this reconciliation. There are at least two distinctly different types of errors which this procedure should detect: (1) payment of an incorrect amount by the customer, and (2) an attempt to post the transaction to the wrong account. A diagram of this one control-two error case appears in Exhibit II.

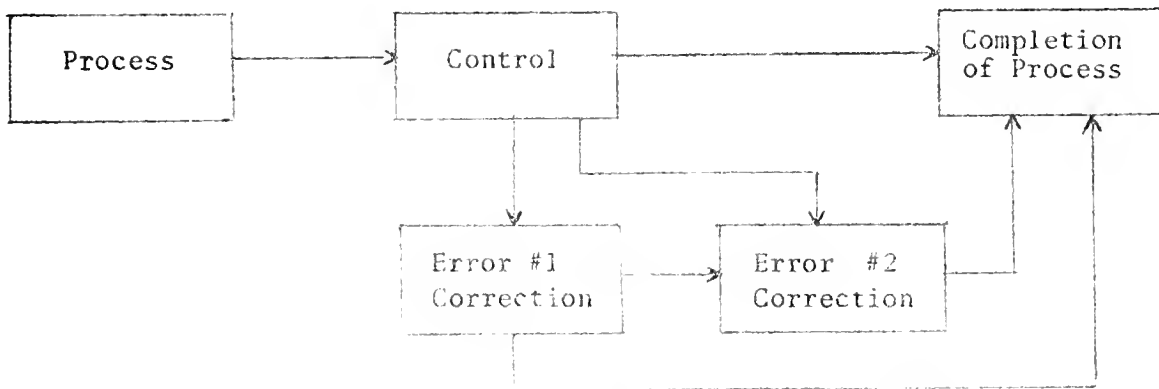


Exhibit II

The development below covers the more general case of n types of errors which may be detected and signalled by a control technique. It is assumed that the error correction procedure is different for each type of error. It is also assumed that the values of all probability parameters

relating to the i th error type are independent of whether any other type of error is present. For example, the probability of detecting and correcting an attempt to post to the wrong account is independent of whether the customer paid the wrong amount. This assumption results in considerable simplification of the analysis.

In a situation of this class, the five basic probability parameters may be expressed as p_i , $P(e_i)$, $P(s_i)$, $P(c_i)$ and $P(d_i)$, where the subscript identifies the particular type of error. Given these parameters, the reliability of the system with respect to each type of error may then be expressed

$$R_i = p_i P(s_i) + p_i (1 - P(s_i)) P(d_i) + (1 - p_i) P(e_i) P(c_i)$$

where R_i represents the probability of completing the process with the i th error not present. The overall reliability, R , or probability of completing the process with no errors of any kind, is then expressed as

$$R = (R_1)(R_2) \dots (R_n)$$

On the other hand, if no control technique is applied, the probability of completing the process with no errors of any kind is

$$p = (p_1)(p_2) \dots (p_n)$$

In this case the difference $R - p$ represents the contribution of the control technique to the overall reliability of the system.

The analysis of costs in this situation is more interesting because C_c , the cost of performing the control technique, is a joint cost, whereas the other cost parameters, C_s and C_e , are unique to each type of error. Therefore, the expected total cost of errors in the process if the control

step is not performed is

$$C_t = \sum_{i=1}^n (1-p_i) C_{e_i}$$

where C_{e_i} represents the average cost of an undetected error of type i .

Alternatively, expected total cost if the control is performed becomes

$$C_t' = C_c + \sum_{i=1}^n (1-R_i)C_{e_i} + \sum_{i=1}^n [p_i(1-P(s_i)) + (1-p_i)P(e_i)]Cs_i$$

where C_c is the cost of performing the error procedure, and Cs_i is the cost of searching for the i th type of error once its existence has been signalled, and of correcting it.

To illustrate these concepts numerically, consider the two-error case where the following set of hypothetical parameter values have been estimated:

p_1	=	.8	p_2	=	.9
$P(e_1)$	=	.95	$P(e_2)$	=	.96
$P(s_1)$	=	.9	$P(s_2)$	=	.95
$P(c_1)$	=	.99	$P(c_2)$	=	.99
$P(d_1)$	=	.99	$P(d_2)$	=	1.0
C_c	=	1	Cs_2	=	2
Cs_1	=	3	C_{e_2}	=	8
C_{e_1}	=	10			

Table 2 summarizes the computations for the formulae presented above. In this example, use of the control technique increases overall reliability by about 26% and reduces expected total costs by .52232 units. In the example involving reconciliation of individual remittances to subsidiary ledger accounts, this figure would represent the expected average cost

reduction per remittance. Multiplication by average daily volume would then be necessary in order to convert to a more meaningful daily cost figure.

$$R_i = p_i P(s_i) + p_i (1 - P(s_i)) P(d_i) + (1 - p_i) P(e_i) P(c_i)$$

$$R_1 = (.8)(.9) + (.8)(.1)(.99) + (.2)(.95)(.98) = .9854$$

$$R_2 = (.9)(.95) + (.9)(.05)(1.0) + (.1)(.96)(.99) = .99504$$

$$R = (R_1)(R_2) = (.9854)(.99504) = .9805 \text{ (approx.)}$$

$$p = (p_1)(p_2) = (.8)(.9) = .72$$

$$R - p = .9805 - .72 = .2605$$

$$\begin{aligned} Ct &= \sum_{i=1}^n (1 - p_i) C e_i \\ &= (.2)(10) + (.1)(8) \\ &= 2.8 \end{aligned}$$

$$\begin{aligned} Ct' &= Cc + \sum_{i=1}^n (1 - R_i) C e_i + \sum_{i=1}^n [p_i (1 - P(s_i)) + (1 - p_i) P(e_i)] C s_i \\ &= 1 + (.0146)(10) + [(.8)(.1) + (.2)(.95)](3) + (.00496)(8) \\ &\quad + [(.9)(.05) + (.1)(.96)](2) \\ &= 1 + .146 + .81 + .03968 + .282 \\ &= 2.27768 \end{aligned}$$

Table 2

Internal Control Systems

Internal control systems comprise one or more internal control techniques directed at one or more errors within a single process or set of related processes. Systems including only one technique are covered in the previous section. This section covers systems of two or more techniques. The first part within the section deals with multiple techniques directed at controlling a single error while the second part deals with systems of multiple controls directed at multiple errors.

Multiple Control-Single Error Case

The simplest form of this type of situation would be a system having two controls, as diagramed in Exhibit III. To provide a realistic example, consider again the use of a batch total to prevent posting of an inaccurate total of cash receipts to the accounts receivable control account. Control #1 would consist of the checking of the batch total by the accounts receivable clerk immediately upon completion of the posting process. A second control would consist of a check by the general ledger clerk of the summary data provided by the accounts receivable clerk against a copy of the bank deposit slip provided by the cashier. This second control would at times detect errors that were not caught by the first control step.

Consider now the effect of multiple control steps on the basic parameters of the control model. There should normally be no effect upon p_j , the probability that the initial process is correctly executed. However, the probabilities pertaining to the control technique and to the error

correction procedure should be unique for each control technique. These will be expressed as $P(e_{ij})$, $P(s_{ij})$, $P(c_{ij})$ and $P(d_{ij})$, where the value of j identifies the particular control technique. The development below covers the general case in which r control techniques are being used.

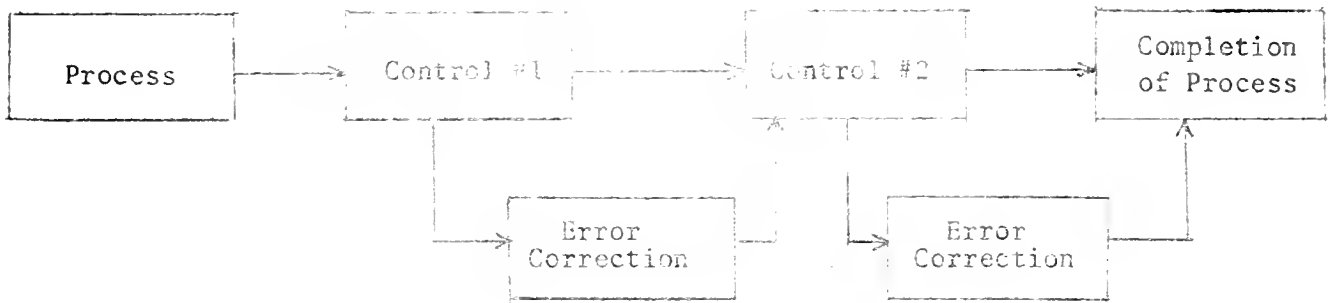


Exhibit III

Among the cost parameters, the cost of performing the control technique is obviously unique to each control technique, and will be expressed as Cc_j . However, the other cost parameters, Cs_i and Ce_j , should be invariant with respect to the control technique used.

To understand the expression of overall reliability in situations involving multiple control techniques, note that the probability that no error is present prior to the second and all subsequent control steps is equal to the reliability of the system up to and including the previous control step. This intermediate reliability is denoted by R_i^j , which, to be precise, represents the probability that no errors of type i are present subsequent to the performance of the j th control step, and of its corresponding error correction procedure where necessary.

Having made this clear, the formula for overall system reliability where r control techniques are used to control a single type of error may now be presented. Assume that the error type is identified by $i = 1$. The expression for R is then as follows:

$$R = R_1^{r-1} P(s_{1r}) + R_1^{r-1} (1-P(s_{1r}))P(c_{1r}) + (1-R_1^{r-1})P(e_{1r})P(c_{1r})$$

As implied by this formula, it is necessary to derive a value for all R_1^j where $0 < j < r$ before R can be computed. The formula for R_1^j has the same form:

$$R_1^j = R_1^{j-1} P(s_{1j}) + R_1^{j-1} (1-P(s_{1j}))P(c_{1j}) + (1-R_1^{j-1})P(e_{1j})P(c_{1j})$$

Of course, $R_1^{j-1} = p_1$ where $j = 1$. Given a particular sequence of performance of control techniques, it is possible to compute the marginal contribution to system reliability of the j th control technique as $R_1^j - R_1^{j-1}$.

With respect to system cost, the expected total cost if no control technique is used is expressed in exactly the same manner as in the single control case, i.e.

$$C_t = (1-p_1) C_{e1}$$

However, if r control techniques are used, the expected average total cost is expressed

$$C_t = \sum_{j=1}^r C_{e_j} + (1-R)C_{e1} + \sum_{j=1}^r \left[R_1^{j-1} (1-P(s_{1j})) + (1-R_1^{j-1})P(e_{1j}) \right] C_{s1}$$

For decision purposes, C_t should be computed for all possible values of r

in order that the decision-maker may decide whether to use all available control techniques or some subset thereof.

Once again these concepts should be clarified by a numerical example. Consider the two-control case where the following set of hypothetical parameter values have been estimated:

$P_1 = .8$	$P(e_{11}) = .9$	$P(e_{12}) = .95$
$Cc_1 = 2$	$P(s_{11}) = .9$	$P(s_{12}) = .96$
$Cc_2 = 3$	$P(c_{11}) = .95$	$P(c_{12}) = .98$
$Ce_1 = 20$	$P(d_{11}) = .99$	$P(d_{12}) = 1.0$
$Cs_1 = 4$		

The various computations applying the formulae presented above are summarized in Table 3. It is interesting to note from the example that the first control step performed adds significantly more to reliability than the second. This is likely to be true in most actual situations. The cost figures in this example also reflect this characteristic, since the total expected average cost using no controls is less than that using two controls, but greater than that using only one control.

$$R_1^j = R_1^{j-1} P(s_{1j}) + R_1^{j-1} (1-P(s_{1j}))P(d_{1j}) + (1-R_1^{j-1})P(e_{1j}) P(c_{1j})$$

$$R_1^1 = (.8)(.9) + (.8)(.1)(.99) + (.2)(.9)(.95) = .97 \text{ (approx.)}$$

$$R = R_1^2 = (.97)(.96) + (.97)(.04)(1.0) + (.03)(.95)(.98) = .99793$$

$$R_1^1 - p_1 = .97 - .8 = .17$$

$$R_1^2 - R_1^1 = .99793 - .97 = .02793$$

$$C_t = (1-p_1)C_{e_1} = (.2)(20) = 4$$

$$\begin{aligned} C_t^1 &= \sum_{j=1}^r C_{c_j} + (1-R) C_{e_1} + \sum_{j=1}^r \left[R_1^{j-1} (1-P(s_{1j})) + (1-R_1^{j-1}) P(e_{1j}) \right] C_{s_1} \\ &= 2 + 3 + (.00207)(20) + \left[(.8)(.1) + (.2)(.9) \right] 4 + \left[(.97)(.04) + (.03)(.95) \right] 4 \\ &= 6.3505 \end{aligned}$$

If only the first control is used:

$$\begin{aligned} C_t^1 &= 2 + (.03)(20) + \left[(.8)(.1) + (.2)(.9) \right] 4 \\ &= 3.64 \end{aligned}$$

Table 3

Multiple Control-Multiple Error Case

This is the general case, and is characteristic of most actual situations where internal controls are used. As an example within the framework of cash receipts processing, consider control techniques which would tend to prevent both (1) posting of an inaccurate total of cash received, and (2) embezzlement of cash receipts. Two possible control techniques would be (1) separation of the functions of the cashier and accounts receivable clerk, and (2) periodic preparation of a bank reconciliation by a person not connected with cash receipts processing. Both of these controls should contribute to the prevention of both of the objectionable conditions cited. A third control, the use of a batch total check, would tend to prevent inaccurate postings, but might contribute little or nothing to the prevention of embezzlement, since the embezzler could always manipulate the batch total.

This example points up the fact that there are many cases where a particular control is not intended to prevent a particular type of error. In these cases, $P(e_{ij}) = 0$ and therefore $R_i^j = R_i^{j-1}$.

The simplest example of a situation in this general class is one involving two control techniques and two errors. Exhibit IV illustrates such a situation in which both controls are directed at both types of errors.

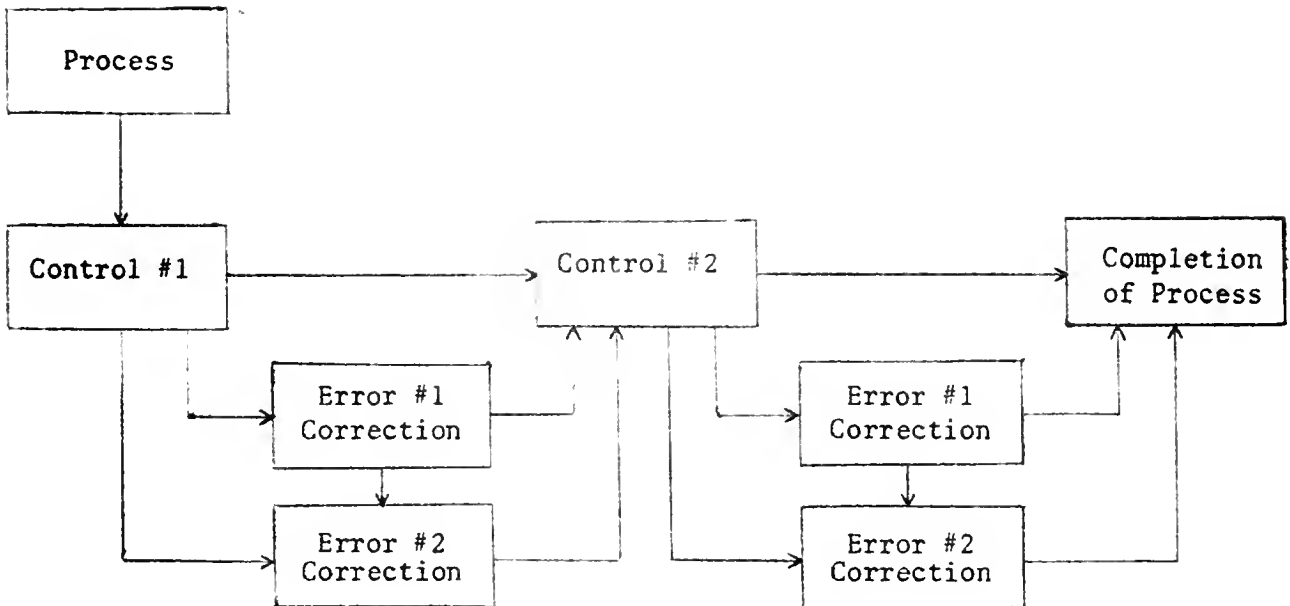


Exhibit IV

The mathematical expression of reliability in the multiple control-multiple error case is a straightforward extension of the two prior cases. The formula for reliability with respect to the i th error at the completion of the j th control step is as follows:

$$R_i^j = R_i^{j-1} P(s_{ij}) + R_i^{j-1} (1-P(s_{ij})) P(d_{ij}) + (1-R_i^{j-1}) P(e_{ij}) P(c_{ij})$$

As before, $R_i^{j-1} = p_i$ where $j = 1$. Where r control techniques are used, R_i^r represents system reliability with respect to the i th error type. Overall system reliability, or the probability that no errors of any kind are present subsequent to the last control step, is expressed

$$R = (R_1^r) (R_2^r) \dots (R_n^r)$$

Expected total cost if no controls are present is expressed in precisely the same manner as in the single control-multiple error case, i.e.

$$C_t = \sum_{i=1}^n (1-p_i) C e_i$$

The formula for expected average total cost where r control techniques are used and n types of errors may be present is as follows:

$$C_t' = \sum_{j=1}^r C c_j + \sum_{i=1}^n (1-R_i^r) C e_i + \sum_{i=1}^n \sum_{j=1}^r \left[R_i^{j-1} (1-P(s_{ij})) + (1-R_i^{j-1}) P(e_{ij}) \right] C s_i$$

As mentioned previously, C_t' should be computed for all possible subsets of the r available control techniques. In this way, the decision-maker can determine whether it is most economical to use all r control techniques, some subset thereof, or no controls at all.

A numerical example should again prove useful in clarifying the various concepts outlined above. Consider the following set of hypothetical parameter values for a two control-two error case:

$p_1 = .5$	$P(e_{11}) = .8$	$P(e_{12}) = .9$
$p_2 = .8$	$P(s_{11}) = .9$	$P(s_{12}) = .94$
$C c_1 = 1$	$P(c_{11}) = .9$	$P(c_{12}) = .96$
$C c_2 = 2$	$P(d_{11}) = .96$	$P(d_{12}) = .98$
$C e_1 = 20$	$P(e_{21}) = .95$	$P(e_{22}) = 0$
$C e_2 = 10$	$P(s_{21}) = .96$	$P(s_{22}) = 1.0$
$C s_1 = 3$	$P(c_{21}) = .98$	
$C s_2 = 2$	$P(d_{21}) = .98$	

Note that $P(e_{22}) = 0$, which means that the second control technique cannot detect the presence of an error of the second type. As a result, $P(s_{22})=1.0$ and $P(c_{22})$ and $P(d_{22})$ are undefined. Table 4 summarizes the computations for the various formulae presented in this section.

Notice in this example that the process under control is only 50% accurate with respect to errors of type 1. In a situation of this type, even the second control contributes substantially to reliability - that is, $R_1^2 - R_1^1 = .12166$. However, in this particular example, the system is slightly more economical with only the first control than it is with both controls. With no controls, expected average total cost is greater than twice as much as with one or both controls.

$$R_i^j = R_i^{j-1} P(s_{ij}) + R_i^{j-1} (1-P(s_{ij})) P(d_{ij}) + (1-R_i^{j-1}) P(e_{ij}) P(c_{ij})$$

$$R_1^1 = (.5)(.9) + (.5)(.1)(.96) + (.5)(.8)(.9) = .858$$

$$R_2^1 = (.8)(.96) + (.8)(.04)(.98) + (.2)(.95)(.98) = .98556$$

$$R_1^2 = (.858)(.94) + (.858)(.06)(.98) + (.142)(.9)(.96) = .97966 \text{ (approx.)}$$

$$R_2^2 = R_2^1 = .98556 \text{ (since } p(e_{22}) = 0)$$

$$R = (R_1^2)(R_2^2) = (.98556)(.97966) = .9655 \text{ (approx.)}$$

$$p = (p_1)(p_2) = (.5)(.8) = .4$$

$$R-p = .9655 - .4 = .5655$$

$$Ct = \sum_{i=1}^n (1-p_i) C e_i$$

$$= (.5)(20) + (.2)(10) = 12$$

$$Ct' = \sum_{j=1}^r C c_j + \sum_{i=1}^n (1-R_i^r) C e_i + \sum_{i=1}^n \sum_{j=1}^r \left[R_i^{j-1} (1-P s_{ij}) + (1-R_i^{j-1}) P(e_{ij}) \right] C s_i$$

$$= 1 + 2 + (.02034)(20) + (.01444)(10) + [(.5)(.1) + (.5)(.8)] 3$$

$$+ [(.858)(.06) + (.142)(.9)] 3 + [(.8)(.04) + (.2)(.95)] 2$$

$$+ [(.98556)(0) + (.01444)(0)] 2$$

$$= 5.883 \text{ (approx.)}$$

If only the first control is used:

$$Ct' = 1 + (.142)(20) + (.01444)(10) + [(.5)(.1) + (.5)(.8)] 3 + [(.8)(.04) + (.2)(.95)] 2$$

$$= 5.7784$$

Extensions of the Model

The set of models presented above represents the basic framework the controller's decision situation with respect to an internal control problem. A controller may use the model (1) to analyze the reliability economic efficiency of an existing internal control system, and/or to optimize the design of a new internal control system.

The auditor may use this same framework to develop internal control recommendations for his clients. However, the auditor's primary decision with respect to internal control involves evaluating the internal control system to assess the extent of audit testing necessary to formulate an opinion on the financial statements. This decision requires a somewhat different version of the model, a version which is developed in this section. Prior to the discussion of this problem, however, two other simple extensions of the model are presented.

Effect of Controls on Process Probabilities

Thus far, an implicit assumption has been made that the presence or absence of a control procedure has no effect upon p_i , the probability of completing the original process without committing an error of type i . However, there are many realistic situations in which this assumption is seriously invalid. For example, the primary contribution of the control technique of separation of functions is likely to be to reduce the probability that embezzlement will ever happen, rather than to increase the

probability of detecting and correcting it once it does happen. The degree of supervision used might have a similar effect. Even the use of a control total might increase the care with which a clerk posts remittances to an accounts receivable ledger. In situations of this type, the presence of the control tends to inhibit the commission of the error.

This factor may be quite easily incorporated into the basic model. All that is required is the definition of a new parameter, p_i^r , which represents the probability that the i th error will not occur in the initial process given that the set of r control techniques are used. Normally, $p_i^r \leq p_i^{r-1}$, which means that error i becomes increasingly less likely to occur as more control techniques are used. To incorporate this parameter into the model, note that $R_i^{j-1} = p_i^r$ when $j = 1$.

Differences in the Timing of Controls

Another simplifying assumption made in the multiple control case is that, in a series of control techniques directed at a particular type of error, the timing of performance of each control is consistent across all of the controls. There are many cases in which this assumption also does not square with reality. For example, both the use of a batch control total and the preparation of a bank reconciliation are directed at errors in posting the total of cash receipts to the general ledger. However, the batch control total may be used once a day, while the bank reconciliation may be prepared once a month. Where differences in timing of this sort exist, the basic model cannot be used without modification.

Assume that the process under study is performed k times prior to the application of the j th control technique. Then the system reliability prior to performance of the j th control step should be expressed $(R_i^{j-1})^k$ instead of simply R_i^{j-1} .

Though the adjustment of reliability is relatively simple, situations of this type also complicate the analysis of costs. Note that both Cs_i , the cost of search, detection and correction of an error, and Ce_i , the cost of an undetected error, should vary directly with the number of errors of type i . If the initial process which begets error i is performed several times prior to the control step, then several errors of type i might be present when the control step is performed.

To explore briefly the effects of this situation upon the cost analysis, consider first the cost of search, detection and correction of an error. This should now be expressed as Cs_{ij} , which represents the average cost of search, detection and correction of an error or errors of type i signalled by the j th control step. A formula for this variable is

$$Cs_{ij} = (Cs_i) (ns_{ij})$$

where Cs_i would represent the cost of search, detection and correction for one error of type i , and ns_{ij} represents the average number of errors of type i signalled by the j th control step. ns_{ij} is a function of R_i^{j-1} and k , and may be easily derived.

In a similar vein, the cost of one or more errors of type i which go undetected by a set of r control steps may be represented by Ce_{ir} , which

could then be expressed

$$Ce_{ir} = (Ce_i)(Ne_{ir})$$

where Ce_i is the average cost of one undetected error of type i, and Ne_{ir} is the average number of errors of type i which go undetected by a set of r control steps. Ne_{ir} may also be derived in terms of the other parameters of the model. Since this derivation is relatively complex and is not essential to a basic understanding of the model, it is not presented here.

Use of the Model in Auditing

To the auditor an accurate measure of R would be useful to the evaluation of an internal control system. However, of even greater value would be an estimate of the expected dollar amount by which each financial statement account balance varies from its "true" balance. Such an estimate may be derived by means of an extension of the basic reliability model. Assume that the estimated total effect of errors of type i on the dollar balance of the account in question is represented by A_i . Then

$$A_i = (Ne_{ir})(Ve_i)(T_r)$$

Where Ne_{ir} is defined as per the previous section, Ve_i represents the average dollar effect of a single undetected error of type i on the balance of the account, and T_r represents the number of cycles of performance of the set of r control techniques over the relevant period of time. To clarify the meaning of T_r , consider that a bank reconciliation is often performed once a month and usually represents the last control step in a cycle of controls relating to the cash account. For a period of one year, then, the value of T_r would be 12 with respect to the set of controls over cash.

Since several different errors may affect the balance of a particular account, the expected total variance of an account balance from its true balance is expressed

$$A = \sum_{i=1}^m \sigma_i^2$$

where the set of errors from $i = 1$ to m encompasses all types of errors which might affect the balance of the account.

It should be noted that A is an expected value, which means that it represents the mean of a range of possible values. An estimate of the standard deviation around this mean value would also prove useful to the auditor. To derive this, estimates of the standard deviations around Ne_{ir} and Ve_j are necessary for each of the m types of errors affecting the account balance. These estimates may easily be developed from data available to the auditor. Following this, the standard deviation around A_i may be derived for each of the n errors, which in turn makes it possible to compute the standard deviation around A .

The auditor will wish to derive an estimate of reliability for each related set of internal controls within an organization, as well as estimates of the mean and standard deviation of the expected difference between each financial statement account balance and its "true" balance. An analysis of this sort will provide an auditor with some objectively derived evidence upon which to base his judgements regarding the extent of testing necessary to the audit, thereby helping him satisfy the second standard of field work.

Problems of Implementation

Any proposal of a new approach to an old problem is probably not complete without a discussion of problems of implementation. In the case of the models presented in this paper, problems of implementation will certainly be significant, but should not be overwhelming.

The basic implementation problem with the models developed in this paper is the derivation of estimates of the probability and cost parameters. However, if a structured program of collection and analysis of past error and cost data is developed, data upon which to base such estimates should be readily available. Consider first the probability parameters upon which the calculation of system reliability is based. Estimates of all of these parameter values - p_i^r , $P(e_{ij})$, $P(s_{ij})$, $P(c_{ij})$ and $P(d_{ij})$ - can be developed from (1) records of error frequencies and error correction procedures maintained by clerical personnel who perform the control procedures, and (2) data collected by internal or external auditors as a normal part of the audit process. Once a procedure for compiling these data is in use, the matter of implementing the basic reliability model should be quite simple.

Consider next the cost parameters. Of these, Ce_j is a clerical cost and should therefore be directly proportional with the amount of time spent on the j th control technique. Similarly, Cs_{ij} should be directly proportional to the average amount of time spent searching for, detecting and correcting the i th error signalled by the j th control step. However, the cost of each type of error, Ce_i , can at best only be estimated on a

subjective basis. Does this mean that the analysis of system costs can be no more accurate than a subjective estimate? The answer is no, because, as explained below, the analysis of costs can be performed without having to estimate specific values for the various Ce_j .

To illustrate the truth of this statement, consider a case where a controller is trying to decide whether or not to use a particular control technique. Given estimates of the five basic probability parameters, and of Cc_j and Cs_{ij} , for all relevant values of i and j , it is possible to solve for a break-even value of Ce_i . This would be the value of Ce_i at which total expected system costs with the control technique are exactly equal to total expected system costs without the control technique. Given this break-even value, the controller then must decide only whether he feels that the actual cost of the error is greater than the break-even cost (in which case the control technique should be used), or less than the break-even cost (in which case the control technique should not be used). Therefore, any choice among alternative control systems may be made on a basis which does not require the making of subjective estimates of error costs.

Implementation of this model by the external auditor requires, in addition to estimates of the five basic probability parameters for all i and j , a measure of Ve_j for each type of error, and of the standard deviation around Ve_i . Since Ve_i represents the average dollar effect of an error of type i on the financial statement account balance, these estimates should be easily prepared from data readily available to the auditor.

It can only be concluded that implementation of the approach outlined in this paper does not present any insurmountable obstacles. Indeed, the primary problem with the models is not likely to be one of deriving the estimates or performing the calculations, but of interpreting the results. The numbers generated by the models may project a sense of certitude which is inconsistent with the fact that all of the model calculations are based upon estimates. It must be recognized by controllers and auditors that the models are intended only to supplement their judgement, not to replace it

Conclusions

The internal control models presented in this paper provide a framework for the analysis and design of internal control systems by controllers and auditors. The models themselves identify the parameters which are relevant to such analysis and design. It therefore seems appropriate that future research on this subject be directed at the problems of developing estimates of these parameters and using the models in a real situation. This suggests that the performance of a field study should perhaps be the next step in future research.

At least one aspect of the models presented herein suggests a possible pragmatic application of behavioral research methodology. Recall the discussion of the potential capacity of a control procedure to inhibit the initial occurrence of an error. Behavioral research could perhaps identify under what conditions and to what extent this is likely to occur.

A third potential avenue of future research would involve extending the models themselves to incorporate the concept of utility. Since a decision not to use a particular control is inherently more risky than its counterpart, a conservative person might reasonably decide to adopt a control which is not economical. Conversely, a less conservative person might decide not to use a control that is economical. Perhaps this explains why some clients fail to adopt control recommendations provided by their auditors.



